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ON IGNORABLE COORDINATES OF CONSERVATIVE AND NATURAL SYSTEMS WITH THREE DEGREES OF FREEDOM*

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A constructive solution of the existence problem for implicit ignorable coordinates** in holonomic systems with 3 degrees of freedom is presented. The solution reduces to the construction of functionally independent Lagrangian invariants of the system in explicit form. Because of the well-known interdependence between the existence of ignorable coordinates in a system and the existence of Killing vector fields, as well as linear (in velocity) first integrals of the Lagrangian equation, a solution of the remaining problems is thereby obtained in a local formulation for systems with 3 degrees of freedom.

1. By definition /1/, the Lagrangian of a natural system has the form

$$L = {}^1/_2\, a_{ij}q^{i\cdot}q^{j\cdot} + a_iq^{i\cdot} + a \quad q^\cdot = dq/dt) \tag{1.1}$$

the coefficients a_{ij}, a_i, and a are functions of the local coordinates $q^1, \ldots, q^n$ and of time t, while $\|a_{ij}\|$ is a positive definite matrix. In formula (1.1) and in subsequent formulas, we will understand the presence of identical indices encountered in the monomials first as superscripts, and then as subscripts, as a sum over all values of the index from 1 to n, with n the number of degrees of freedom of the system. A natural system is said to be invertible /2/ if the vector $b = (a_1,\ldots, a_n)$ is zero, and is not invertible otherwise. The configuration manifold X_n of a system is assumed to be C^r-smooth, and the functions a_{ij}, a_i, and a, with $i, j = 1,\ldots, n$, do not contain t explicitly and belong to the class C^r over X_n (the number r will be assumed to be sufficiently great, so that the existence and continuity of all the derivatives of the functions used in further discussions will not be expressly stated).

Suppose U is some coordinate neighborhood of an arbitrary point $p \in X_n$. We consider the following problem: given the Lagrangian coefficients (1.1), it is necessary to determine whether the function $\Psi(q^1, \ldots, q^n)$ and the nondegenerative transformation of coordinates

$$q^j = q^j(Q^1,\ldots, Q^n)\ (j = 1, \ldots, n) \tag{1.2}$$

exists in U, such that the Lagrangian

$$L + \frac{d}{dt}\Psi = {}^1/_2 A_{ij}Q^{i\cdot}Q^{j\cdot} + A_iQ^{i\cdot} + A \tag{1.3}$$

has ignorable coordinates (for the case of invertible systems $\Psi(U) = 0$).

In the case $n = 2$ the solution of the problem is given by theorem 1.

We introduce the notation

$$V_2 = (X_2,\ ds^2 = a_{ij}dq^idq^j),\ \ \delta^2 = a_{11}a_{22} - (a_{12})^2,\ \ \partial_i = \partial/\partial q^i$$
$$\operatorname{rot} b = (\partial_1a_2 - \partial_2a_1)/\delta,\ \ \Delta_1f = a^{ij}\partial_if\partial_jf,\ \ \Delta_2f = \partial_i(\delta a^{ij}\partial_jf)/\delta,$$
$$\|a^{ij}\| = \|a_{ij}\|^-$$

The last formulas constitute the definitions of the differential parameters Δ_1f and Δ_2f of the function $f(q^1,\ldots, q^n)$, which are valid for arbitrary n.

Let $\Phi(q^1, q^2) \in C^2$ is some function defined in U and without any critical point. Then $p \in X_n$ is said to be a critical point of the function $f\colon X_n \to R$, if $df(p) = 0$.

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** Editor's Note: The ignorable coordinates are also known as the cyclic coordinates.

Theorem 1. 1) If

$$\partial_1 F\partial_2\Phi - \partial_1\Phi\partial_2 F = 0 \ (F = a, \ \mathrm{rot}\ b, \ \Delta_1\Phi, \ \Delta_2\Phi) \tag{1.4}$$

then local coordinates may be defined on set U, one of which is an ignorable coordinate for a given conservative natural system with 2 degrees of freedom.

2) An ignorable coordinate also exists whenever the functions a, rot b, and K (K is the Gaussian curvature of the Riemannian manifold) are constant in U; the two coordinates of the system may be simultaneously ignorable if and only if $K(U) = (\mathrm{rot}\ b)(U) = 0$ and $a(U) =$ const.

Obviously, if an ignorable coordinate exists for any of the functions a, rot b, or K, taken as the Φ, the conditions (1.4) will be satisfied. The sufficiency of the latter conditions is proved by introducing the new coordinates Q^1 and Q^2, and by selecting the function Ψ by the formulas

$$Q^1 = \Phi(q^1, q^2), \quad Q^2 = \mu(a_{21}\partial_1\Phi - a_{11}\partial_2\Phi)\,dq^1 + \mu(a_{22}\partial_1\Phi - a_{12}\partial_2\Phi)\,dq^2$$

$$\mu = \frac{1}{\delta}\exp\left(-\int\frac{\Delta_2\Phi}{\Delta_1\Phi}\,d\Phi\right), \quad \Psi = -\int B_1\,dQ^1 + (B_2 - A_2)\,dQ^2$$

$$a_i\,dq^i = B_i\,dQ^i, \quad A_2 = \Delta\ \mathrm{rot}\ b\,dQ^1, \quad \Delta^2 = A_{11}A_{22} - (A_{12})^2$$

In the expression

$$L + \frac{d}{dt}\Psi = \frac{1}{2\Delta_1 Q^1}\left[(Q^{1\cdot})^2 + \frac{(Q^{2\cdot})^2}{\mu^2\delta^2}\right] + A_2(Q^1)\,Q^{2\cdot} + A(Q^1)$$

the coordinate Q^2 is ignorable. If $K(U) = \mathrm{const}$, there exists /3,4/ coordinates Q^1 and Q^2, such that $ds^2 = (dQ^1)^2 + A_{22}(Q^1)(dQ^2)^2$, hence follows the second assertion of the theorem.

The theorem includes as special cases the classical attribute /3,4/ of a rotation metric, existence criterion /5–7/ for an ignorable coordinate in invertible systems, and an analogous criterion /8/ for noninvertible systems. Below we will find a solution of the problem for the case $n = 3$.

Definition. The tensor field $y_{\alpha\beta\ldots\tau}(q^1, \ldots, q^n)$ defined on the Riemannian manifold $V_n = (X_n, ds^2 = a_{ij}dq^i dq^j)$ is said to be an invariant of the Lagrangian (1.1) if:

1) the components $y_{\alpha\beta\ldots\tau}$ are functions only of the components a_{ij} of the metric tensor, the skew-symmetric quantities $\omega_{lk} = \partial_l a_k - \partial_k a_l$ $(1 \leqslant l < k \leqslant n)$, the scalar function a and the partial derivative to some order of these arguments with respect to $q^1, \ldots, q^n$:

$$y_{\alpha\beta\ldots\tau} = f_{\alpha\beta\ldots\tau}\left(a_{ij}, \frac{\partial a_{ij}}{\partial q^r}, \ldots, \frac{\partial^m a_{ij}}{\partial^p q^1 \ldots \partial^h q^n};\right.$$

$$\left.\omega_{lk}, \frac{\partial\omega_{lk}}{\partial q^c}, \ldots, \frac{\partial^s\omega_{lk}}{\partial^x q^1\ldots\partial^z q^n}; a, \frac{\partial a}{\partial q^b}, \ldots, \frac{\partial^e a}{\partial^v q^1\ldots\partial^u q^n}\right)$$

(m, s, and e are fixed nonnegative integers; $p, \ldots, h, x, \ldots, z, v, \ldots, u$ run through all possible nonnegative integers, such that $p + \ldots + h = m$, $x + \ldots + z = s$, $v + \ldots + u = e$, and the remaining indices run through the values $1, 2, \ldots, n$);

2) if (1.2) and (1.3) are expressed in terms of arbitrary local coordinates, the new components

$$Y_{\alpha\beta\ldots\tau} = f_{\alpha\beta\ldots\tau}\left(A_{ij}, \frac{\partial A_{ij}}{\partial Q^r}, \ldots, \frac{\partial^m A_{ij}}{\partial^p Q^1\ldots\partial^h Q^n};\right.$$

$$\left.\Omega_{lk}, \frac{\partial\Omega_{lk}}{\partial Q^c}, \ldots, \frac{\partial^s\Omega_{lk}}{\partial^x Q^1\ldots\partial^z Q^n}; A, \frac{\partial A}{\partial Q^b}, \ldots, \frac{\partial^e A}{\partial^v Q^1\ldots\partial^u Q^n}\right)$$

$$\left(\Omega_{lk} = \frac{\partial A_k}{\partial Q^l} - \frac{\partial A_l}{\partial Q^k}, \ 1 \leqslant l < k \leqslant n, \ A = a\right)$$

where A_{ij} are the new components of the metric tensor.

Remark 1°. For the purposes of the present paper, it is not necessary for the second condition to hold for all possible transformations of coordinates. The transformation (1.2) may be limited by the condition $\mathrm{sgn}\det\|\partial_i Q^j\| = 1$. Therefore it is assumed that the quantities $y_{\alpha\beta\ldots\tau}$ will be components of the pseudo-tensor, i.e., coefficients of the external differential form. Henceforth, this distinction will not be expressly stated.

2°. Whenever the invariant has null rank and is independent of the functions a and ω_{ij}, and their partial derivatives, we arrive at the classical definition /3/ of a scalar differential invariant of the manifold V_n (Gaussian invariant). If the invariant has null rank, and is independent of the components ω_{ij} and their partial derivatives, it is called /3/ the scalar differential parameter of the function a (Beltrami invariant).

3°. All possible covariant derivatives of order r $\nabla_{\alpha\beta\ldots\tau}a$ of the function a obviously form the invariant of the Lagrangian (1.1) of arbitrary rank r.

4°. If $n=3$ and $h(h_1, h_2, h_3)$ is a differentiable vector function over the manifold X_3, then, as is well known /3,5/, the quantities

$$\beta^1 = \frac{1}{\delta}(\partial_2 h_3 - \partial_3 h_2), \quad \beta^2 = \frac{1}{\delta}(\partial_3 h_1 - \partial_1 h_3), \tag{1.5}$$
$$\beta^3 = \frac{1}{\delta}(\partial_1 h_2 - \partial_2 h_1), \quad \delta^2 = \det \| a_{ij} \|$$

in an arbitrary coordinate system $\{q^1, q^2, q^3\}$ define the contravariant components of the pseudovector, denoted $\operatorname{rot} h$. If $\operatorname{sgn}\det\|\partial_i Q^j\| = 1$ is restricted to the transformation (1.2), $\omega = \operatorname{rot} b$ will be the vector invariant of the Lagrangian (1.1).

5°. By definition, it follows that if there exists $2 \leqslant k \leqslant n$ functionally and mutually independent scalar invariants of Lagrangian (1.1) (if $k=1$, the invariant may not be a constant), than for any selection of the local coordinates and function Ψ, the Lagrangian (1.3) cannot have more than $n-k$ ignorable coordinates.

Further, we set $n=3$. Let $da \neq 0$ at the point p. Then there exists a neighborhood U in which the solution of the equation $a = a(x)$ for every point $x \in U$ is a two-dimensional submanifold. In other words, for all values of the constant c in some interval $I \subset R$, the equation $a=c$ defines a single-parameter family of surfaces $\{a=c\}$ in U, such that one and only one surface of the family passes through every point $x \in U$. If the vector invariant y of the Lagrangian satisfies the condition $y \times \operatorname{grad} a \neq 0$, in U, the field

$$y - \frac{y \cdot \operatorname{grad} a}{\Delta_1 a}\operatorname{grad} a$$

is a nonzero invariant and its unit vector

$$e_1 \in \bigcup_{c \in I}\bigcup_{x \in U} T_x(a=c) \tag{1.6}$$

where $T_x(a=c)$ is a tangent space at the point x on the surface $a=c$. The invariant $e_2 = (\Delta_1 a)^{-1/2} e_1 \times \operatorname{grad} a$ is said to be complementary to e_1, and also satisfies the inclusion relation (1.6).

Fields of identity vectors tangent to the lines of curvature of surfaces of the family $\{a=c\}$ are nontrivial examples of the invariants e_1 and e_2.

Let us prove this assertion in the particular case in which the surfaces in the family $\{a=c\}$ are geodetically parallel (this is the only case which will be necessary below). A functional relation between the invariants $\Delta_1 a$ and a may be used /5/ as an analytic criterion of such a (relative position of the surfaces. Without loss of generality, we may assume that $\Delta_1 a = 1$. If the surfaces are geodetically parallel, the components z^1, z^2, z^3 of the vector e_k yield an extremal value for the form $E = {}^1/_2 \nabla_{ij} a z^i z^j$ at every fixed point $x \in U$ under the conditions $\nabla_i a z^i = 0$ and $a_{ij} z^i z^j = 1$ /6/. We generate the auxiliary function

$$\Pi = E + {}^1/_2 \lambda (a_{ij} z^i z^j - 1) + \mu \nabla_i a z^i$$

and then write down the necessary extremum conditions

$$(\lambda a_{ij} + \nabla_{ij} a) z^j + \mu \nabla_i a = 0 \ (i = 1, 2, 3)$$

If z^1, z^2, z^3 is a solution of the problem, we will have

$$\mu = -(\nabla_{ij} a \nabla^i a + \lambda \nabla_j a) z^j = -\left(\frac{1}{2}\frac{\partial \Delta_1 a}{\partial a} + \lambda\right)\nabla_j a z^j = 0$$

Consequently, the factor λ must be a root of the equation

$$\delta^{-2}\det\|\lambda a_{ij} + \nabla_{ij} a\| = \lambda^3 + H\lambda^2 + K_{\text{rel}}\lambda + \delta^{-2}\det\|\nabla_{ij} a\| = 0 \tag{1.7}$$
$$H = \Delta_2 a, \ K_{\text{rel}} = {}^1/_2\, e^{ijk} e^{lmv} a_{il} \nabla_{jm} a \nabla_{kv} a$$

Here, $e^{ijk} = 1/\delta$ $(e^{ijk} = -1/\delta)$, if the sequence i, j, k is obtained by even (odd) permutation of the indices $1, 2, 3$; $e^{ijk} = 0$ otherwise. The quantities $H(x)$ and $K_{\text{rel}}(x)$ are the values at $x \in U$

corresponding to the mean and relative curvature /3/ of the surface $a = a(x)$. All the roots of equation (1.7) are real. One of the roots is zero. Thus, we have verified quite simply that by selecting a semigeodetic coordinate system $v^1, v^2, v^3 = a$, in which the linear element of the manifold V_3 is written in the form /3/

$$ds^2 = b_{\alpha\beta} dv^\alpha dv^\beta + (da)^2 \; (\alpha, \beta = 1, 2) \tag{1.8}$$

while the matrix of the second covariant derivatives of a has the form

$$\nabla_{ij} a = \begin{Vmatrix} {}^1/_2 \partial_a b_{11}, & {}^1/_2 \partial_a b_{12}, & 0 \\ {}^1/_2 \partial_a b_{21}, & {}^1/_2 \partial_a b_{22}, & 0 \\ 0, & 0, & 0 \end{Vmatrix}$$

If the other roots $\lambda_1 \neq \lambda_2$, we may associate with each root λ_k $(k = 1,2)$ obviously a unique solution $e_k(z^1, z^2, z^3)$ of the equations

$$(\lambda_k a_{ij} + \nabla_{ij} a) z^j = 0, \; \nabla_i a z^i = 0, \; a_{ij} z^i z^j = 1 \; (i = 1, 2, 3) \tag{1.9}$$

and $e_1 e_2 = 0$. Thus, the quantities z^j are expressed functionally in terms of the components $a_{ij}, \nabla_{ij} a, \nabla_i a$. Equations (1.9) are obtained in an arbitrary local coordinate system $\{q^1, q^2, q^3\}$, so that e_1 and e_2 in fact are invariants of the Lagrangian (1.1), Q.E.D.

2. If the function $a(x)$ is not constant, the system may have 2,1, or no single ignorable coordinate. Let us analyze these case in turn.

Theorem 2. 1) Let that the function $f \in C^2$ is defined in a neighborhood $U \ni p$ and does not have any critical points

$$e_1 \in \bigcup_{c \in I} \bigcup_{x \in U} T_x (f = c)$$

is a doubly continuously differentiable unit vector of some field, and e_2 is the unit vector of the complementary field. If the expressions

$$a, \; \Delta_1 f, \; \pi_{\alpha\beta} = e_\alpha \operatorname{rot} e_\beta, \; \omega e_\alpha \; (\alpha, \beta = 1, 2) \tag{2.1}$$

are functions only of f and, in addition,

$$\omega \cdot \operatorname{grad} f = 0 \tag{2.2}$$

then in a sufficiently small neighborhood of the point $p \in X_3$, it is possible to select generalized coordinates of the system, two of which are ignorable.

2) if there exist two ignorable coordinates in the system simultaneously and if f is some nontrivial scalar Lagrangian invariant, any other scalar invariant is a function only of f, and condition (2.2) is satisfied.

Proof. The second assertion follows obviously from the definition in Sect.1. In the coordinate system $\{Q^1, Q^2, Q^3\}$, where Q^1, Q^2 are ignorable coordinates of the Lagrangian (1.3), the vector $\Omega = \omega$ has by (1.5) the components

$$\Omega^1 = -\partial_3 A_2 / \Delta, \; \Omega^2 = \partial_3 A_1 / \Delta, \; \Omega^3 = 0$$
$$(\Delta^2 = \det \| A_{ij} \|, \; \operatorname{grad} A = (0, 0, \partial_3 A))$$

Since $f = f(Q^3)$, the condition (2.2) is satisfied.

Let us now prove the first assertion of the theorem. We consider the equations

$$dq^i / ds = a^{ij} \partial_j f \; (i = 1, 2, 3) \tag{2.3}$$

of the normal congruence of the family $\{f = c\}$. To define an individual curve in the congruence, we need only specify the coordinates of the intersection point of this curve and some fixed surface in the family. Let us select the surface $f = f(p)$ and define the coordinate system $\{\gamma^1, \gamma^2\}$ with coordinate vectors e_1, e_2 on it. In a sufficiently small neighborhood U of the point $p \in X_3$, the general solution of equations (2.3) may be represented in the form

$$v^1(q^1, q^2, q^3) = \gamma^1, \; v^2(q^1, q^2, q^3) = \gamma^2$$

where v^1 and v^2 are regular functions. Obviously, the functions $v^1, v^2, v^3 = f$ are independent in U and may be selected as the new coordinates of the system. In these coordinates, the linear element

$$ds^2 = b^{\alpha\beta} dv^\alpha dv^\beta + b_{33} (df)^2 \; (\alpha, \beta = 1, 2)$$

Since $b_{33} = 1/(\Delta_1 f)$ is a function only of f, without loss of generality we may assume that the linear element is written in the form (1.8).

Let us prove that the coefficients b_{11}, b_{12} and b_{22} do not contain v^1 and v^2. By condition, we have

$$b^{\alpha\beta} w_{\lambda\alpha} w_{\iota\beta} = \varepsilon_{\lambda\iota}, \quad e^{\alpha\beta} w_{\lambda\alpha} \frac{\partial w_{\iota\beta}}{\partial f} = \pi_{\lambda\iota}\delta \quad (\alpha, \beta, \lambda, \iota = 1, 2) \tag{2.4}$$

where $\| b^{\alpha\beta} \| = \| b_{\alpha\beta} \|^{-1}$ $(w_{\lambda 1}, w_{\lambda 2}, 0)$ are the covariant components of the vector e_λ in the semigeodetic coordinate system $\{v^1, v^2, f\}$, $\varepsilon_{11} = \varepsilon_{22} = 1$, $\varepsilon_{12} = \varepsilon_{21} = 0$, $e^{11} = e^{22} = 0$, $e^{21} = -e^{12} = 1$, $\delta^2 = b_{11}b_{22} - (b_{12})^2$. Since $w_{11}w_{22} - w_{12}w_{21} \neq 0$, we find from equations (2.4) that

$$b_{11} = w_{12}^2 + w_{22}^2, \quad b_{12} = w_{11}w_{12} + w_{21}w_{22}, \quad b_{22} = w_{11}^2 + w_{21}^2 \tag{2.5}$$

$$\partial w_{\lambda\alpha}/\partial f = e^{\iota\beta}\pi_{\iota\lambda} w_{\beta\alpha} \quad (\lambda, \alpha = 1,2) \tag{2.6}$$

Thus, the components of the vectors e_1 and e_2 satisfy equations (2.6) and the further condition

$$w_{11} = 1, \ w_{12} = 0, \ w_{21} = 0, \ w_{22} = 1 \quad \text{if } f = f(p) \tag{2.7}$$

But, by the existence and uniqueness theorem for differential equations, the solution of the system (2.6) under the initial condition (2.7) is a function only of f. Consequently, the coefficients (2.5) are independent of the coordinates v^1, v^2. Q.E.D.

As a corollary, we find that

$$\operatorname{grad} f \cdot \operatorname{rot} e_\lambda = 0 \quad (\lambda = 1, 2) \tag{2.8}$$

These relations have a simple geometric meaning. If Γ is some smooth curve in a two-dimensional Riemannian manifold and if $\tau\,(\tau_1, \tau_2)$ is the unit vector of its tangent, the pseudoscalar $\operatorname{rot} \tau = (\partial_1\tau_2 - \partial_2\tau_1)/\delta$ will equal the geodetic curvature of the curve /9/. An arbitrary surface $f = c$ defines the Riemannian submanifold $V_2 = (x \in U{:}\ f(x) = c,\ ds^2 = b_{\alpha\beta}dv^\alpha dv^\beta;\ \alpha, \beta = 1, 2)$.

The relations (2.8) state that smooth curves on the surface $f = c$, which at every one of its points are tangent to one of the vectors e_1 or e_2, are geodetic lines of the submanifold V_2.

The ignorable coordinates of the metric $ds^2 = a_{ij}dq^i dq^j$ may be found in the following way. We consider the relation

$$\nu\,(\mu e_1 + e_2) = \operatorname{grad} Q\,(q^1, q^2, q^3) \tag{2.9}$$

The left side of this equality is a gradient if and only if /10/ $(\mu e_1 + e_2) \operatorname{rot} (\mu e_1 + e_2) = 0$, i.e.,

$$\partial\mu/\partial f + \mu^2\pi_{11} + \mu\,(\pi_{21} + \pi_{12}) + \pi_{22} = 0 \tag{2.10}$$

The coefficients of the resulting equation depend only on f, so that it may be considered as an ordinary Riccati-type differential equation, with μ the unknown function and f the independent variable.

Let $\mu_1(f) \neq \mu_2(f)$ are any two particular solutions of equation (2.10). With every solution $\mu_k\ (k = 1, 2)$, we associate some function $\nu_k\,(q^1, q^2, q^3) \neq 0$, such that the curl of the left side of (2.9) is equal zero. Multiplying it as a scalar by the noncoplanar vectors $\mu_k e_1 + e_2$, e_1, and $\operatorname{grad} f$, we obtain equation (2.10) and, in light of (2.8), the equation

$$\partial\nu_k/\partial f - \nu_k\,(\mu_k\pi_{11} + \pi_{12}) = 0, \quad \operatorname{grad} \nu_k \cdot \operatorname{grad} f \times (\mu_k e_1 + e_2) = 0$$

respectively. Consequently $\nu_k = \nu_k(f)$. The independent functions $Q^k\ (k = 1, 2)$ obtained by squaring equation (2.9) under the condition $\mu = \mu_k$, $\nu = \nu_k$ and $Q^3 = f$, may be taken as the new coordinates of the system. In these coordinates

$$ds^2 = A_{11}\,(dQ^1)^2 + 2A_{12}dQ^1dQ^2 + A_{22}\,(dQ^2)^2 + (dQ^3)^2$$
$$A_{11} = \frac{(\mu_2)^2 + 1}{(\nu_1)^2(\mu_1 - \mu_2)^2}, \quad A_{12} = -\frac{\mu_1\mu_2 + 1}{\nu_1\nu_2(\mu_1 - \mu_2)^2} \quad A_{22} = \frac{(\mu_1)^2 + 1}{(\nu_2)^2(\mu_1 - \mu_2)^2}$$

Thus, the coefficients of the homogeneous quadratic part of the Lagrangian (1.1) related to the coordinates Q^1, Q^2, and Q^3, are independent of Q^1 and Q^2.

By condition , we have for the linear part of the Lagrangian (1.1) $a_i q^{i\cdot} = B_i Q^{i\cdot}$

$$\frac{\partial B_1}{\partial Q^2} = \frac{\partial B_2}{\partial Q^1}, \quad \frac{\partial B_1}{\partial Q^3} = \frac{\partial B_3}{\partial Q^1} + R\,(Q^3)\,\Delta, \quad \frac{\partial B_2}{\partial Q^3} = \frac{\partial B_3}{\partial Q^2} + W\,(Q^3)\,\Delta \tag{2.11}$$
$$(R = \nu_2\omega\,(\mu_2 e_1 + e_2), \quad W = \nu_1\omega\,(\mu_1 e_1 + e_2), \quad \Delta^2 = \det \| A_{ij} \|)$$

These relations, understood as equations in B_1, B_2 and B_3, admit of a partial solution of the form $A_i = A_i(Q^3)$ $(i = 1, 2, 3)$. Consequently $B_i = A_i + \partial_i\Phi(Q^1, Q^2, Q^3)$ $(i = 1, 2, 3)$. We set $\Psi = -\Phi$, thereby obtaining the Lagrangian (1.3) in which all the coefficients are functions of the single coordinate Q^3. The theorem is proved.

Remark 6°. The conditions of the theorem help in discovering the presence of two implicit ignorable coordinates in the system or to prove that such coordinates do not exist. For example, suppose that $da(p) \neq 0$. Then in some neighborhood $U \ni p$, we may set $f = a$. In a non-invertible system, it is convenient to set

$$e_1 = \omega\,(a_{ij}\omega^i\omega^j)^{-1/2}$$

whenever condition (2.2) is satisified (otherwise, by the second assertion of the theorem, the system will not have two ignorable coordinates).

If $H^2 - 4K_{\mathrm{rel}} \neq 0$ at the point $p \in X_3$, in some neighborhood U of p we may take as the e_1 and e_2 (invariants) the unit tangent vectors to the lines of curvature of surfaces in the family $\{a = c\}$.

7°. Suppose that the conditions of Theorem 2 do not hold. That is, the expressions (2.1) and $P = \omega\cdot\mathrm{grad}\,f$ are functions of the single variable f, but $P \neq 0$. Then the Lagrangian system reduces to a form with a single ignorable coordinate. In fact, reasoning in the same way as in the proof of the first assertion of the theorem, we find, not (2.11), but the system of equations

$$\frac{\partial B_1}{\partial Q^2} = \frac{\partial B_2}{\partial Q^1} + P(Q^3)\,\Delta, \quad \frac{\partial B_1}{\partial Q^3} = \frac{\partial B_3}{\partial Q^1} + R(Q^3)\,\Delta, \quad \frac{\partial B_2}{\partial Q^3} = \frac{\partial B_3}{\partial Q^2} + W(Q^3)\,\Delta$$

These equations are compatible only if $P\Delta \equiv \mathrm{const}$, and admit of the particular solution

$$A_1 = Q^2 P\Delta + \int R\Delta\, dQ^3, \quad A_2 = \int W\Delta\, dQ^3, \quad A_3 = 0$$

Consequently $B_i = A_i + \partial_i\Phi(Q^1, Q^2, Q^3$ $(i = 1, 2, 3))$. Setting $\Psi = -\Phi$, we obtain the Lagrangian (1.3) whose coefficients are independent of Q^1.

In light of remark 6°, the following assertion is of interest.

Theorem 3. Let a natural system is invertible and that in some neighborhood $U \ni p$, we have $df \neq 0$ $(H^2 - 4K_{\mathrm{rel}})(f) = 0$. Then:

1) In order that two ignorable coordinates exist in the system simultaneously, it is sufficient, first, that the expressions

$$a,\ \Delta_1 f,\ \Delta_2 f \tag{2.12}$$

be functions of f and, second, that the absolute curvature K_{abs} of the surface $f = f(p)$ vanish, i.e.,

$$\left[\frac{1}{2(\Delta_1 f)^2} e^{ijk} e^{lmr} \partial_i f \partial_l f \nabla_{jm} f \nabla_{kr} f\right]\Big|_{f=f(p)} = 0 \tag{2.13}$$

2) If $f(x)$ is a scalar invariant of the system, both conditions are necessary for the existence of two ignorable coordinates.

Proof. Necessity. The necessity of the first condition is self-evident. Since the metric of the submanifold $V_2 = (x \in U\colon f(x) = f(p);\ ds^2|_{f=f(p)})$ is Euclidean, the absolute (Gaussian) curvature of the surface $f = f(p)$ is zero. Note that the expression in square brackets in (2.13) for K_{abs} follows from the Mashke formula /11/.

Sufficiency. In the semigeodetic coordinates v^1, v^2, f the linear element of the manifold V_3 is written in the form (1.8) (under the condition $\Delta_1 f = 1$). Here $v^1 = \gamma^1$, $v^2 = \gamma^2$ are the general integrals of the equations (2.3), and γ^1, γ^2 are the local coordinates on the surface $f = f(p)$. By condition (2.13), the coordinate system $\{\gamma^1, \gamma^2\}$ may be selected as Cartesian, in which case

$$b_{11} = 1,\ b_{12} = 0,\ b_{22} = 1 \quad \text{if} \quad f = f(p) \tag{2.14}$$

The condition $(H^2 - 4K_{\mathrm{rel}})|_U = 0$ states that at every point $x \in U$, the principal curvatures of the surface $f = f(x)$ coincide, i.e., x is the umbilical point of the surface $f = f(x)$. By definition /3/, at an umbilical point the coefficients of the first and second principal quadratic forms of a surface are proportional:

$$\Omega_{11}/b_{11} = \Omega_{12}/b_{12} = \Omega_{22}/b_{22} \tag{2.15}$$

In the semigeodetic coordinates v^1, v^2, f we have /3/

$$\Omega_{\alpha\beta} = \partial_f b_{\alpha\beta} \ (\alpha,\ \beta = 1,\ 2)$$
$$\Delta_2 f = \frac{b_{22}\partial_f b_{11} + b_{11}\partial_f b_{22} - 2b_{12}\partial_f b_{12}}{2\,(b_{11}b_{22} - b_{12}^2)} \tag{2.16}$$

We find from equations (2.15) and (2.16) that

$$\partial b_{\alpha\beta}/\partial f = b_{\alpha\beta}\Delta_2 f \ (\alpha,\ \beta = 1,\ 2) \tag{2.17}$$

Thus, the coefficients $b_{\alpha\beta}$ satisfy the system (2.17) and the initial condition (2.14). Consequently

$$b_{11} = b_{22} = \exp\Big(\int_{f(p)}^{f} \Delta_2 f\, df\Big),\quad b_{12} = \exp\Big(\int_{f(p)}^{f} \Delta_2 f\, df\Big) - 1$$

The theorem is proved.

Remark 8°. If the natural system is invertible, and if $f(x)$ is some function, such that $df \neq 0$ and $(H^2 - 4K_{\text{rel}})(f) = 0$ in the neighborhood $U \ni p$, the differential parameters $\Delta_1 f,\ \Delta_2 f,\ K_{\text{abs}}$ are functions of f, but $K_{\text{abs}}|_{f=f(p)} \neq 0$, then the Lagrangian of the system reduces to a form with a single ignorable coordinate.

In fact, in this case the absolute curvature of the surface $f = f(p)$ is constant, therefore /3,4/ there exist coordinates γ^1, γ^2, in which

$$ds^2|_{f=f(p)} = G(\gamma^2)\ (d\gamma^1)^2 + (d\gamma^2)^2$$

The solution of equations (2.17) with initial condition $b_{11} = G(\gamma^2),\ b_{12} = 0,\ b_{22} = 1$ has the form $b_{\alpha\beta} = b_{\alpha\beta}(\gamma^2, f) = b_{\alpha\beta}(v^2, f)$, in the case $f = f(p)$, i.e., v^1 is an ignorable coordinate.

3. In view of Remarks 7° and 8°, we may consider the case in which there are two functionally independent invariants in the set of invariants (2.1) ($f,\ \omega\cdot\text{grad}\ f$) or (2.12) ($f,\ K_{\text{abs}}$).

Theorem 4. 1) Suppose that are given certain functions φ_1 and φ_2 defined on the set U and that their differentials are independent at every point. If the expressions

$$\Delta_1\varphi_\alpha,\ \nabla(\varphi_1,\ \varphi_2),\ \text{grad}\ \varphi_\alpha\cdot\text{rot}\ \tau,\ \tau\cdot\text{rot}\ \tau \tag{3.1}$$

$$\omega\cdot\text{grad}\ \varphi_\alpha,\ \ \omega\tau\ \ (\nabla(\varphi_1,\ \varphi_2) = a^{ij}\partial_i\varphi_1\partial_j\varphi_2,\ \ \tau = \text{grad}\ \varphi_1 \times \text{grad}\ \varphi_2;\ \alpha = 1{,}2) \tag{3.2}$$

and a are functions only of φ_1 and φ_2, in a neighborhood $U \ni p$ it is possible to select coordinates of the system one of which will be an ignorable coordinate.

2) If there exists at most a single ignorable coordinate in the system, any pair of functionally and mutually independent scalar invariants φ_1 and φ_2 of the Lagrangian (1.1) will satisfy condition 1.

Proof. Necessity. From the definition in Sect.1, it follows that any scalar invariant of the Lagrangian (1.1) is a function of positional coordinates. Therefore, the conditions of the theorem must be satisfied, since the invariants φ_1 and φ_2 are independent.

Sufficiency. Let us prove that there exists functions $f(\varphi_1,\ \varphi_2) \neq 0$ and $h(\varphi_1,\ \varphi_2)$, such that

$$\lambda f = \text{grad}\ Q\ (q^1,\ q^2,\ q^3) \tag{3.3}$$
$$(\lambda = h\varkappa + \tau,\ \varkappa = e^{\alpha\beta}\,\nabla(\varphi_1,\ \varphi_\beta)\cdot\text{grad}\ \varphi_\alpha;\ \alpha,\ \beta = 1,\ 2)$$

In place of the coordinates q^1, q^2 and q^3, we take the coordinates $\psi, \varphi_1, \varphi_2$, where $\psi(q^1, q^2, q^3)$ satisfies in U the condition

$$\frac{D(\psi, \varphi_1, \varphi_2)}{D(q^1, q^2, q^3)} \neq 0$$

The left side of (3.3) is a gradient if and only if /10/ $\lambda\ \text{rot}\ \lambda = 0$, i.e.,

$$\frac{\partial h}{\partial\psi}\ \tau\cdot(\text{grad}\ \psi \times \varkappa) + he^{\alpha\beta}\nabla(\varphi_1, \varphi_\beta)\,\text{grad}\ \varphi_\alpha\cdot\text{rot}\ \tau +$$
$$\frac{\partial}{\partial\varphi_i}\,[h\nabla(\varphi_1, \varphi_i)]\,(\Delta_1\varphi_1\Delta_1\varphi_2 - [\nabla(\varphi_1, \varphi_2)]^2) + \tau\cdot\text{rot}\ \tau = 0\ , \quad (\alpha, \beta, i = 1, 2)$$

Since the expressions (3.1) are functions of φ_1 and φ_2, the resulting equation admits of a solution of the form $h = h(\varphi_1, \varphi_2)$. With this solution we may associate the function $f(\varphi_1, \varphi_2, \psi) \neq 0$, such that the curl of the left side of (3.3) is zero:

$$\text{grad}\ f \times \lambda + f\ \text{rot}\ \lambda = 0 \tag{3.4}$$

Here

$$\mathrm{rot}\,\lambda = \tau \frac{\partial}{\partial \varphi_i}[h\nabla(\varphi_1, \varphi_\tau)] + \mathrm{rot}\,\tau \qquad (i = 1, 2)$$

Equality (3.3) defines some function Q for the given functions h and f. Further, we assume that $\psi = Q$.

The linear element of the manifold V_3 referred to the coordinates $Q^1 = \psi$, $Q^2 = \varphi_1$, $Q^3 = \varphi_2$, is given by

$$ds^2 = A_{ij}dQ^i dQ^j,\ \| A_{ij} \| = \| A^{ij} \|^{-1} \tag{3.5}$$
$$A^{11} = f^2 \ (h^2\Delta_1\varphi_1 + 1)\ (\Delta_1\varphi_1\Delta_1\varphi_2 - [\nabla(\varphi_1, \varphi_2)]^2), \quad A^{12} = 0$$
$$A^{13} = fh\ (\Delta_1\varphi_1\Delta_1\varphi_2 - [\nabla(\varphi_1, \varphi_2)]^2),\ A^{22} = \Delta_1\varphi_1, \quad A^{23} = \nabla(\varphi_1, \varphi_2),\ A^{33} = \Delta_1\varphi_2$$

Consequently, the determinant of the quadratic form (3.5) is expressed thus $\Delta^2 = M(\varphi_1, \varphi_2) f^{-2}$.

The vector equation (3.4) is written in an equivalent form if it is multiplied as a scalar by given noncoplanar vectors grad φ_1, grad φ_2, and λ. We obtain a system of two equations, and the third equation is satisfied identically by virtue of our choice of h. This system

$$\frac{1}{f\Delta}\cdot\frac{\partial f}{\partial \varphi_2} + f\,\mathrm{grad}\,\varphi_1\cdot\mathrm{rot}\,\tau = 0, \qquad \frac{1}{f\Delta}\cdot\frac{\partial f}{\partial \varphi_1} - f\,\mathrm{grad}\,\varphi_2\cdot\mathrm{rot}\,\tau = 0$$

is compatible and, obviously, admits of a nonzero solution of the form $f(\varphi_1, \varphi_2)$. Q.E.D.

Let us consider the components of the vector $\omega = \mathrm{rot}\,b$ in the coordinate system $\{Q^1, Q^2, Q^3\}$:

$$\frac{1}{\Delta}\left(\frac{\partial B_3}{\partial Q^2} - \frac{\partial B_2}{\partial Q^3}\right) = \omega\,(he^{\alpha\beta}\nabla(\varphi_1, \varphi_\beta)\,\mathrm{grad}\,\varphi_\alpha + \tau)\,f \tag{3.6}$$
$$\frac{1}{\Delta}\left(\frac{\partial B_1}{\partial Q^3} - \frac{\partial B_3}{\partial Q^1}\right) = \omega\cdot\mathrm{grad}\,\varphi_1$$
$$\frac{1}{\Delta}\left(\frac{\partial B_2}{\partial Q^1} - \frac{\partial B_1}{\partial Q^2}\right) = \omega\cdot\mathrm{grad}\,\varphi_2, \quad B_i Q^{i\cdot} = a_i q^{i\cdot}$$

From (3.6), we find that

$$\partial/\partial\varphi_i\,(\Delta\omega\cdot\mathrm{grad}\,\varphi_i^{\cdot}) = 0 \ \ (i = 1, 2) \tag{3.7}$$

Since the functions (3.2) depend only on φ_1 and φ_2 and since condition (3.7) is satisfied, the equality (3.6), understood as a system of eqations in B_1, B_2 and B_3, obviously admits of a particular solution of the form $A_i = A_i\,(Q^2, Q^3)$ $(i = 1, 2, 3)$. Consequently $B_i = A_i + \partial_i\Phi\,(Q^1, Q^2, Q^3)$ $(i = 1, 2, 3)$. Setting $\Psi = -\Phi$, we obtain the Lagrangian (1) with cyclic coordinate Q^1. The theorem is proved.

4. If a conservative natural system possesses the invariant $v\,(q^1, q^2, q^3)$, such that the condition $dv \neq 0$ in a sufficiently small neighborhood $U \ni p$, the preceding results allow us to construct a solution for the existence problem of implicit ignorable coordinates of the system in U. The function $v = a$ is the simplest scalar invariant of the Lagrangian (1.1). Since the function a is continuously differentiable, such a neighborhood exists whenever $da\,(p) \neq 0$.

If p is a critical point of the function a, two cases are possible, depending upon whether the following condition is or is not satisfied. There exists a neighborhood U, in which $a\,(V) \neq \mathrm{const}$ for any open subset $V \subset U$.

The first case does not present any difficulty. Note an assertion valid for arbitrary $n \geqslant 2$: if the solution of the equation $a = a\,(p)$ is not a submanifold in U, in no neighborhood of p may we introduce a coordinate system with $(n-1)$-ignorable coordinates (otherwise the equation $a = a\,(p)$ would define a coordinate surface, i.e., a submanifold in U).

In the second case, we may use our results if instead of a, we find some other scalar invariant β of the Lagrangian (1.1), such that $d\beta\,(p) \neq 0$. Then in the statement of the first assertion of Theorem 2, we must set $f = \beta$. The scalar invariance of the vector field $\omega = \mathrm{rot}\,b$ and the Gaussian invariants of the manifold V_3 are the scalar invariants of the Lagrangian (1.1). It is useful to bear in mind that an explicit form of the second- and third-order Gaussian invariants have been presented /12,13/ for a differential quadratic form with three independent variables.

The second case occurs whenever the Lagrangian

$$L = {}^1/_2\ a_{ij}q^{i\cdot}q^{j\cdot} + a_i q^{i\cdot} \tag{4.1}$$

In a system with the Lagrangian (4.1), all three coordinates may be ignorable. The existence criterion for a Euclidean metric /3/ implies the following assertion.

Theorem 5. Three ignorable coordinates can exist simultaneously in a system with the Lagrangian (4.1) if and only if $\omega = 0$ and if the six essential components of the curvature tensor of the manifold V_3 are zero.

5. The problem of determining the ignorable coordinates of a conservative natural system is also equivalent to the problem of determining certain particular intervals of the system. Let us consider a natural family of trajectories of the system /14/, i.e. ∞^{2n-1} family of solutions of the Lagrange equations corresponding to a single general value of the constant h of the energy integral. The integral

$$f_i(q^1, \ldots, q^n, h)\, q^{i\cdot} + f(q^1, \ldots, q^n, h) = \text{const} \tag{5.1}$$

of this family is called /2/ the conditional linear integral of the Lagrange equations, and its coefficients may contain h as a parameter. In addition to this system, which has the Lagrangian (1.1), we introduce an auxiliary system with configuration manifold $X_n^* = \{x \in X_n : h + a > 0\}$ and Lagrangian

$$L^* = {}^1/_2 (h + a)\, a_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} + a_i \frac{dq^i}{d\tau} \tag{5.2}$$

where h is a parameter and τ is an independent variable.

The following assertion is true:

Theorem 6. If relation (5.1) expresses the conditional integral of a given system corresponding to the value h of the constant of the energy integral, the relation

$$(h + a) f_i \frac{dq^i}{d\tau} + f = \text{const} \tag{5.3}$$

is a general integral of the auxiliary system.

Proof. We set $d\tau = (h + a)\, dt$, expression (5.1) turns into (5.3), and the Lagrange equations of the system assume the form

$$\frac{d}{d\tau}\frac{\partial L^*}{\partial q^{j'}} - \frac{\partial L^*}{\partial q^j} + \left(\frac{1}{2} a_{ik} q^{i'} q^{k'} - \frac{1}{h + a}\right)\frac{\partial a}{\partial q^j} = 0, \qquad (j = 1, \ldots, n), \quad \left(q' = \frac{dq}{d\tau}\right)$$

Consequently, the natural family of trajectories with constant h is a family of integral curves of the equations

$$\frac{d}{d\tau}\frac{\partial L^*}{\partial q^{j'}} - \frac{\partial L^*}{\partial q^j} = 0 \quad (j = 1, \ldots, n) \tag{5.4}$$

$${}^1/_2 (h + a)\, a_{ik} q^{i'} q^{k'} = 1 \tag{5.5}$$

We differentiate the integral (5.3) with respect to τ by virtue of equations (5.4), obtaining

$${}^1/_2 (\nabla_i \eta_k + \nabla_k \eta_i)\, q^{i'} q^{k'} + [\partial_k \eta + \eta^i (\nabla_i a_k - \nabla_k a_i)] q^{k'} = 0 \tag{5.6}$$

where $\eta_j = (h + a) f_j$ $(j = 1, \ldots, n)$, and $\eta = f$ are the covariant derivatives computed relative to the metric of the Riemannian manifold $V_n^* = (X_n^*, dl^2 = (h + a)\, a_{ij} dq^i dq^j)$. Equality (5.6) must be satisfied for all possible values of $q^{1'}, \ldots, q^{n'}$ that satisfy the relation (5.5). If $q^{1'}, \ldots, q^{n'}$ is such a set of values, the quantities $-q^{1'}, \ldots, -q^{n'}$ also satisfy (5.5). Therefore the quadratic and linear parts of (5.6) must separately vanish. Hence, using well-known lines of reasoning we find that

$$\nabla_i \eta_k + \nabla_k \eta_i = 0, \; \partial_k \eta_i + \eta^i (\nabla_i a_k - \nabla_k a_i) = 0 \, (i, k = 1, \ldots, n)$$

But these equalities follow from (5.6) whenever the quantities $q^{j\cdot}$ are not related by any formulas. The theorem is proved.

Remark 9°. As a corollary, we may derive that if a system with the Lagrangian (1.1) has a conditional linear integral corresponding to some value of the energy constant h, there exists a transformation of coordinates and function $\Psi(q^1, \ldots q^n)$, such that the transformed Lagrangian $L^* + \Psi'$ has at least one ignorable coordinate. In the case of systems with two degrees of freedom, this assertion was first proved by G.D. Birkhoff /2,15/ by means of isothermal coordinates, and the necessary and sufficient existence conditions for a conditional linear integral obtained in /16/. The remarks in Sect.4 must be used to obtain analogous conditions for systems with three degrees of freedom.

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